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Factorisations of 4-Regular Graphs and Petersen's Theorem

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On the basis of the observation that a 3-regular graph has a perfect matching if and only if its line graph has a triangle-free 2-factorisation, we show that a connected 4-regular graph has a triangle-free 2-factorisation, provided it has no more than two cut-vertices belonging to a triangle. This result is equivalent to Petersen's theorem about the existence of perfect matchings in 3-regular graphs. © 1995 Academic Press, Inc.

1. INTRODUCTION; PRELIMINARIES

One of the classical results about the factorisation of graphs is Petersen's theorem that every regular graph of even degree has a 2-factor (and hence a 2-factorisation) [2]. One may modify the classical problem—which simply asks for the existence of a 2-factor—by imposing bounds on the number or size of the components of the factors, for example, in terms of the degree: given a regular graph G of even degree $d \geqslant 4$, when does G have

(1) a 2-factor consisting of at most n/d cycles, where n is the order of G;

(2) a 2-factor consisting of cycles of length $\geqslant d$;

(3) a 2-factorisation all of whose factors consist of cycles of length $\geqslant d$?

Obviously $(3) \Rightarrow (2) \Rightarrow (1)$. Question (1) in a more general form (G a near-regular graph of arbitrary, not necessarily even, degree containing a 2-factor) was raised by Faudree at the Petersen Centennial Conference [1].

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In this article we shall deal with question (3) in the case $d=4$; i.e., we ask for conditions on a 4-regular graph to have a *triangle-free* 2-factorisation. Examples of 4-regular graphs without such factorisations (indeed, without any triangle-free 2-factor) are quite easy to construct if one allows the existence of cut-vertices which belong to a triangle; we shall call such cut-vertices *essential*. The following theorem provides a broad sufficient condition for the existence of triangle-free 2-factorisations.

1.1. THEOREM. *Any (simple) connected 4-regular graph with at most two essential cut-vertices has a triangle-free 2-factorisation. In particular, this holds for 2-connected 4-regular graphs.*

Taken by itself, this result has the appearance of dealing with a special situation of limited importance. Its main interest lies in the fact that it is equivalent to Petersen's classical theorem about the factorisation of 3-regular graphs.

THEOREM (Petersen). *Any connected 3-regular graph with at most two bridges has a perfect matching.*

In a sense, Theorem 1.1 may be thought of as the "4-regular version" of Petersen's theorem. The link between the two—straightforward in one direction, rather more subtle in the other—is provided by line-graphs (note that the line-graph of a 3-regular graph is 4-regular). The key observation is the following Theorem 1.2.

1.2. THEOREM. *Let G be a (simple) 3-regular graph. Then G has a perfect matching if and only if the line graph $L(G)$ has a triangle-free 2-factorisation.*

Given any graph G, the cut-vertices of $L(G)$ are in one–one correspondence with the bridges of G, and if G has minimum degree $\geqslant 3$, they are all essential. Hence with the help of Theorem 1.2, Petersen's theorem follows immediately from Theorem 1.1.

Going in the opposite direction, we shall prove Theorem 1.1 as a consequence of Petersen's theorem. Here, however, we cannot invoke Theorem 1.2, as Theorem 1.1 deals with a wide class of 4-regular graphs which in general bear no resemblance to line graphs. The idea is to split the given graph G into a *triangular* and a *triangle-free* part, i.e., the union of all triangles of G and its edge-complement, respectively. The two parts are then factorised separately. For the triangle-free part we use a variation of Petersen's method for factoring 4-regular graphs (taking alternate edges in an eulerian walk). The triangular part can be treated much like a line graph, and it is here and in correctly matching the factorisations of the two parts that Petersen's theorem comes into play. This is carried out in Section 4.

For this approach to work it is necessary that both the triangular and the triangle-free part of G be eulerian. In general this is not the case; we eliminate this obstacle by showing that it suffices to prove Theorem 1.1 for graphs whose triangles are pairwise edge-disjoint. This reduction forms the subject of Section 3. Section 2 contains the proof of Theorem 1.2.

For technical reasons (arising primarily from the reduction process in Section 3) it will be convenient to work not only with 4-regular graphs. To describe the wider class of graphs, here are some definitions. All graphs will be *finite* and *simple* unless stated otherwise.

A graph is an *i, j-graph*, where $i > j$ are two given positive integers, if all its vertices are of degree i or j (in particular, i-regular and j-regular graphs are i, j-graphs). A vertex of degree i will be called an *i-vertex* for short.

Of interest to us are 4,2-graphs. Note that if G and H are 4,2-graphs and H is a subgraph of G, then $G \backslash H$ is also a 4,2-graph. (By $G \backslash H$ is meant the *edge-complement* of H in G, i.e., the edge-induced subgraph of G with $E(G \backslash H) = E(G) \backslash E(H)$.)

A *4,2′-graph* is a simple 4,2-graph G which is obtained from a loopless 4-regular graph G_0 (multiple edges permitted) by subdividing every edge of G_0 by an even number (possibly zero) of 2-vertices.

It is immediate from this definition that (i) every 4,2′-graph has an even number of edges; and (ii) no 2-vertex of a 4,2′-graph belongs to a triangle.

A *bicoloration* of a graph G is a partition of $E(G)$ into two (colour) classes ($+1$ and -1). A bicoloration is *balanced* if each vertex of G is balanced, i.e., has the same number of positive and negative incident edges; it is *triangle-free* if G contains no monochromatic triangle. For 4-regular graphs a balanced bicoloration is the same as a 2-factorisation.

We shall prove the following slightly more general form of Theorem 1.1.

1.3. THEOREM. *Any connected 4,2′-graph with at most two essential cut-vertices has a triangle-free balanced bicoloration.*

It should be noted that if multiple edges are permitted, Theorem 1.1 becomes false, even for 2-connected graphs. This is easily seen by taking any 2-connected, 4-regular graph and replacing one of its edges by the graph of Fig. 1. In any 2-factor of the resulting graph G the edge e_0 lies in a triangle.

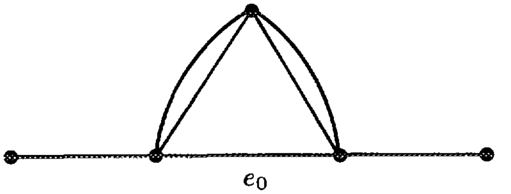

FIGURE 1

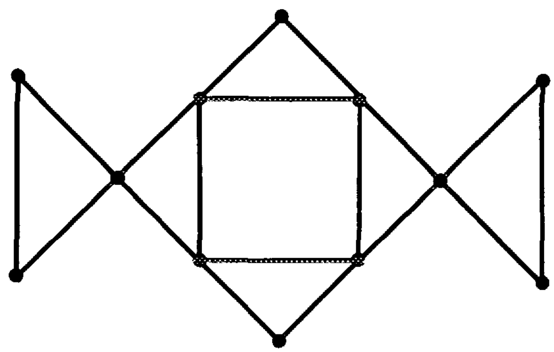

FIGURE 2

Note also that Theorem 1.3 no longer holds if one allows 4,2-graphs with an even number of edges instead of 4,2′-graphs, as witness the graph of Fig. 2 (in which the two triangular blocks can be replaced by any 2-connected 4,2-graph with an odd number of edges). G has balanced bicolorations but none of them is triangle-free.

2. MATCHINGS AND TRIANGLE-FREE 2-FACTORS

In this section we shall prove Theorem 1.2 in the following more formalised version.

2.1. THEOREM. *Given a 3-regular graph G, there is a natural surjection from the set of all triangle-free 2-factorisations of the line graph $L(G)$ to the set of all perfect matchings of G.*

For the proof we need the following.

2.2. LEMMA. *Let G be a 3-regular graph with a perfect matching M. Then there exists a bipartition A, B of $V(G)$ such that* (1) *every edge of M is an AB-edge*; and (2) *every component of the 2-factor $G \backslash M$ meets both A and B. Hence every cycle, in particular every triangle, of G meets both A and B.*

Proof. Choose an edge from each component of $G \backslash M$. These edges, together with M, form a spanning subgraph G' of G. Since G is 3-regular, every cycle in G' is an alternating cycle for M. Thus G' is bipartite; any bipartition A, B of $V(G')$ $(= V(G))$ has the required properties.

If C is any cycle in G, then either C contains an edge of M (which is an AB-edge by (1)), or, as G is 3-regular, C is a component of $G \backslash M$. ∎

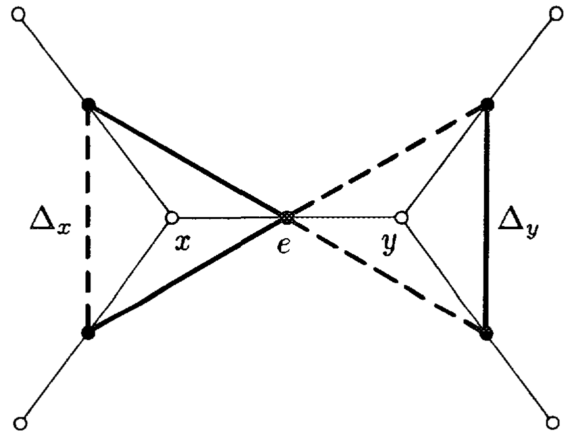

FIG. 3. Triangle-free bicoloration of a double triangle in $L(G)$ (——— $= +1$, - - - - - $= -1$).

Proof of Theorem 2.1. For each $x \in V(G)$ let Δ_x be the "canonical" triangle in $L(G)$ whose vertices are the edges of G incident with x. G being simple, these triangles are pairwise edge-disjoint.

Given a triangle-free 2-factorisation of $L(G)$ it follows that each Δ_x has exactly two edges of equal colour, i.e., belonging to the same factor; let e_x be their common endpoint. Considered as an edge in G, e_x joins x to some vertex y. In $L(G)$, the two edges of Δ_y incident with e_x also have equal colour (opposite to the dominant colour in Δ_x), and therefore $e_y = e_x$. This means that $M = \{e_x : x \in V(G)\}$, considered as a set of edges of G, is a perfect matching.

Conversely, every perfect matching of G can be obtained by the preceding construction. Given a perfect matching M let A, B be a bipartition of $V(G)$ as in Lemma 2.2.

For a given edge $e = [x, y]$ of G denote by D_e the "double triangle" corresponding to e in $L(G)$, i.e., $D_e := \Delta_x \cup \Delta_y$. Since M is a perfect matching, the double triangles D_e, $e \in M$, form an edge-disjoint decomposition of $L(G)$. Taking into account that every $e \in M$ is an AB-edge, we colour the edges of D_e as shown in Fig. 3; i.e., for the endpoint of e in A, say x, the dominant colour of the edges of Δ_x is $+1$ and for the endpoint in B it is -1.

Combining the bicolorations of the D_e's, $e \in M$, we obtain a 2-factorisation of $L(G)$, and it is immediately obvious that none of the canonical triangles is monochromatic. Moreover, since no triangle of G has all its vertices in A or in B, it follows that none of the "non-canonical" triangles (i.e., those induced by the triangles in G) is monochromatic either. Finally, it is clear that if the construction of the first part of the proof is applied to this 2-factorisation, the result is the original matching M. ∎

Observe that if G is a triangle-free, then the only triangles in $L(G)$ are the canonical ones. In this case Lemma 2.1 may be disregarded, because it simply says that one may arbitrarily assign one endpoint of each edge in M to A and the other to B. For the preceding proof this means that the double triangles D_e, $e \in M$, can be coloured independently of each other; i.e., it is irrelevant which of the two triangles Δ_x, Δ_y has $+1$ as its dominant colour.

3. Reduction to Edge-Disjoint Triangles

For the proof of Theorem 1.3 we observe, to begin with, that it suffices to prove the theorem under the additional assumption that all cut-vertices of the given 4,2′-graph G are essential. This is easily seen by induction on the number of inessential cut-vertices of G. Given such a cut-vertex u, consider the two subgraphs G_1, G_2 into which u separates G. In G_i, u is of degree 2; remove u from G_i and connect its two neighbours by a new edge e_i, $i = 1, 2$. The resulting graph G'_i is simple because u does not belong to any triangle of G. Clearly G'_i is a 4,2′-graph, and it has fewer inessential cut-vertices than G. If G'_1 and G'_2 have balanced triangle-free bicolorations we may assume (by reversing the colours in one of the two graphs if necessary) that e_1 and e_2 have opposite colours. The two bicolorations can then be combined to form a bicoloration of G by giving the two edges of G_i incident with u the colour of e_i, $i = 1, 2$. Clearly this bicoloration is balanced and triangle-free.

The following reduction lemma is slightly stronger than what is actually needed for the proof of Theorem 1.3, in that no assumption is made about the number of (essential) cut-vertices.

3.1. Lemma. *Let G be a connected 4,2′-graph all of whose cut-vertices are essential and which contains two triangles sharing an edge. Then there exists a connected 4,2′-graph G' such that* (1) *G' has fewer edges than G;* (2) *G' has the same number of cut-vertices as G, and they are essential;* (3) *if G' has a triangle-free balanced bicoloration, then so does G.*

If in G' there are again triangles with a common edge, then Lemma 3.1 can be applied to G' and this process can be repeated as long as triangles with common edges are present. Since at each step the number of edges decreases, we finally reach a "reduced" 4,2′-graph which, in addition to (1), (2), (3), has the property that its triangles are pairwise edge-disjoint (there may be no triangles at all). In other words, it suffices to prove Theorem 1.3 under the additional hypothesis that no two triangles of the given graph share an edge.

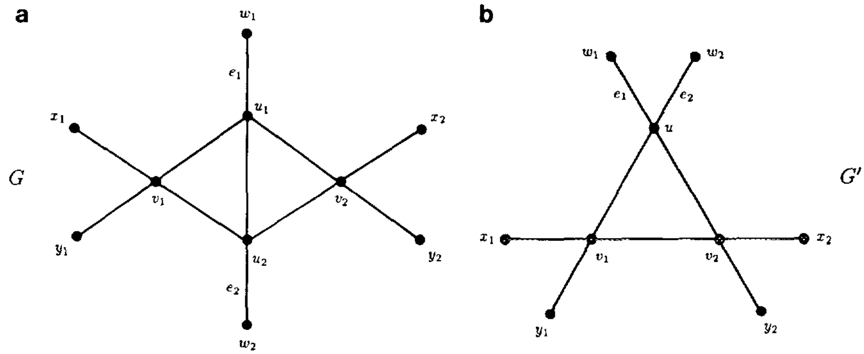

FIGURE 4

It should be noted that Lemma 3.1 does not claim that G' has fewer triangles than G. In fact there is no way to assign to every 4,2'-graph G a 4,2'-graph G' with property (3) and fewer triangles than G. For if such an assignment were possible, then its repeated application would lead to a triangle-free 4,2'-graph and for the latter, all balanced bicolorations are trivially triangle-free. Because of property (3) it would then follow that all 4,2'-graphs have a balanced triangle-free bicoloration, and this is obviously false. In the proof of Lemma 3.1 it can be seen that in certain cases the number of triangles may actually go up. For example, this will happen in the situation described by Fig. 9, provided G contains the edges x_1x_2, x_1y_1, x_1y_2, x_2y_1, x_2y_2.

Proof of Lemma 3.1. No edge of a graph of maximum degree 4 can belong to more than three triangles. We can therefore distinguish the following cases.

Case 1. G has an edge $e=[u_1, u_2]$ which belongs to exactly two triangles $\Delta_i = u_1u_2v_i$, $i=1, 2$, $v_iv_1 \neq v_2$.

Subcase 1.1. v_1, v_2 are non-adjacent. In this case we have the configuration depicted in Fig. 4a, where $w_1 \neq w_2$ (otherwise e is in three triangles), $x_1 \neq y_1$, $x_2 \neq y_2$. Except for these restrictions some of the six vertices $w_1, w_2, x_1, x_2, y_1, y_2$ may coincide. In other words, there may be additional triangles that share an edge with Δ_1 or Δ_2.

Consider the graph G' obtained by replacing $\Delta_1 \cup \Delta_2$ by a single triangle as shown in Fig. 4.b. Clearly G' is a connected 4,2'-graph. Its cut-vertices are those of G and possibly u. The latter is a cut-vertex if and only if $\{e_1, e_2\}$ is a cut of G. Note that the "additional" triangles mentioned earlier (if they exist) remain unaffected by the reduction of G to G'.

Assume first that $\{e_1, e_2\}$ is not a cut of G. Then G' has the same cut-vertices as G, and they are essential in G' as well.

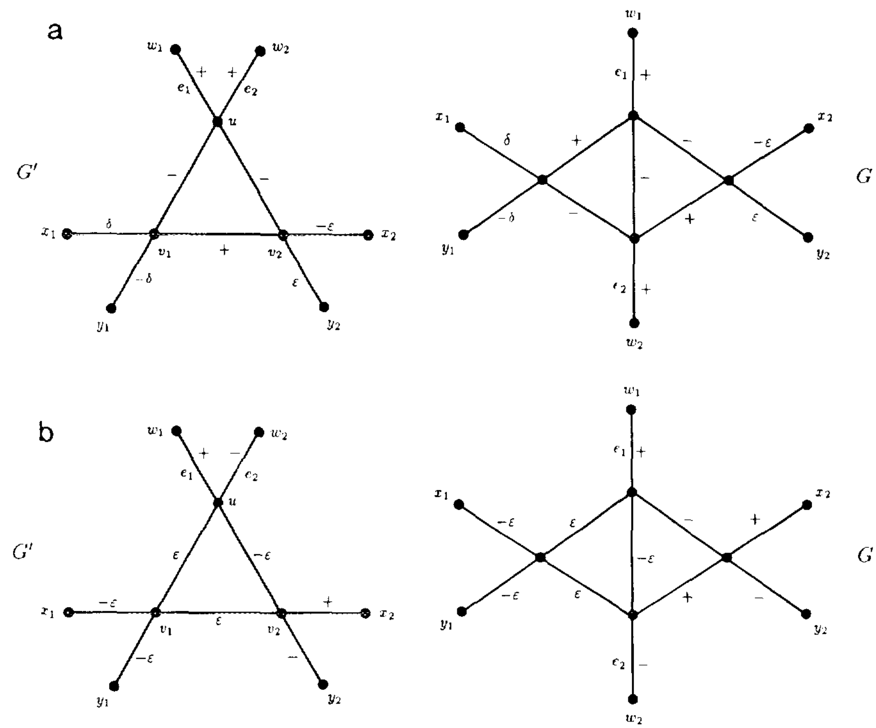

FIGURE 5

Balanced bicolorations of G' are easily seen to induce balanced bicolorations of G. Since we are proceeding from the smaller graph to the larger one we shall speak of "lifting" a bicoloration from G' to G. This lifting is shown in Figs. 5a and 5b for the two cases where e_1 and e_2 have equal (w.l.o.g. $= +1$) or opposite colour.

Assuming the given bicoloration of G' to be triangle-free, it is clear that the lifted bicoloration also is triangle-free except when e_1 and e_2 have equal colour, $x_1 = w_1$ and the triangle $u_1 v_1 w_1$ is monochromatic ($\delta = +1$ in our choice of colours), or $y_2 = w_2$ and $u_2 v_2 w_2$ is monochromatic ($\varepsilon = +1$). Suppose the former. In this case we lift the bicoloration of G' as in Fig. 6. The lifted bicoloration is triangle-free except when $x_2 = w_1$ and $\varepsilon = -1$, or $y_2 = w_1$ and $\varepsilon = +1$. However, this situation cannot occur because it would mean that $v_1 v_2 w_1$ is a monochromatic triangle in G', contrary to the triangle-freeness of the bicoloration of G'.

To complete Subcase 1.1 assume now that $\{e_1, e_2\}$ is a cut of G. Note that this implies that w_1, w_2 cannot be equal to any of x_1, x_2, y_1, y_2.

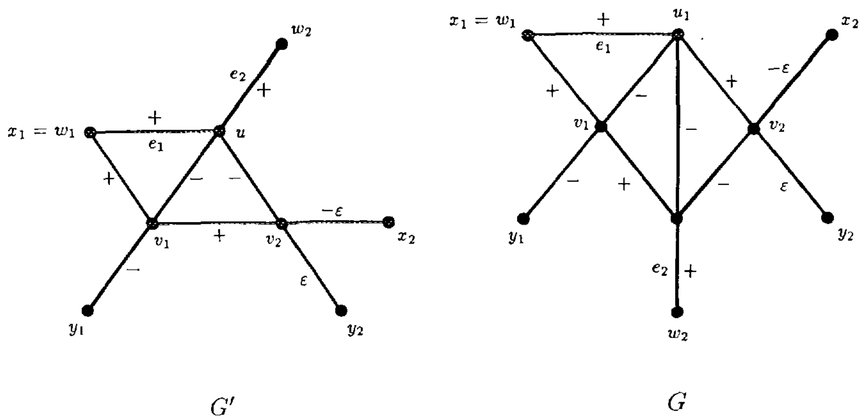

FIGURE 6

We reduce G to a graph G'' as shown in Fig. 7. Clearly G'' is a connected 4,2′-graph, $\{e_1, e_2\}$ is a cut of G'', and the component W of $G''\setminus\{e_1, e_2\}$ containing w_1, w_2 is the same as in G (for degree reasons, w_1, w_2 are in the same component). The cut-vertices of G'' are the same as those of G, and all are essential.

Since G is a 4,2′-graph, the number of edges of W is odd. This means that in any balanced bicoloration of G'', e_1 and e_2 have the same colour. Depending on whether e_0 and e_1 have different or equal colour, bicolorations of G'' are lifted to G as indicated in Figs. 8a and 8b (where $\delta, \varepsilon = +1$ or -1). In the second case, the colours of the edges of W and of e_1, e_2 have to be reversed. Since W is a component of $G\setminus\{e_1, e_2\}$, this reversal creates no monochromatic triangles.

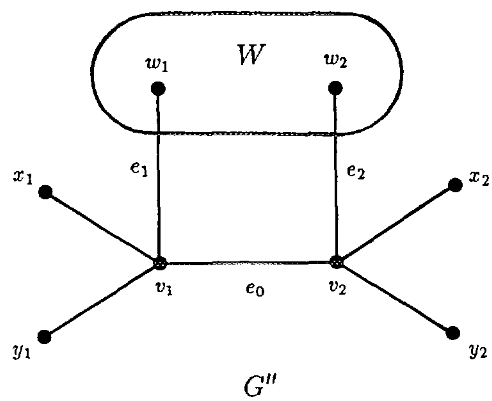

FIGURE 7

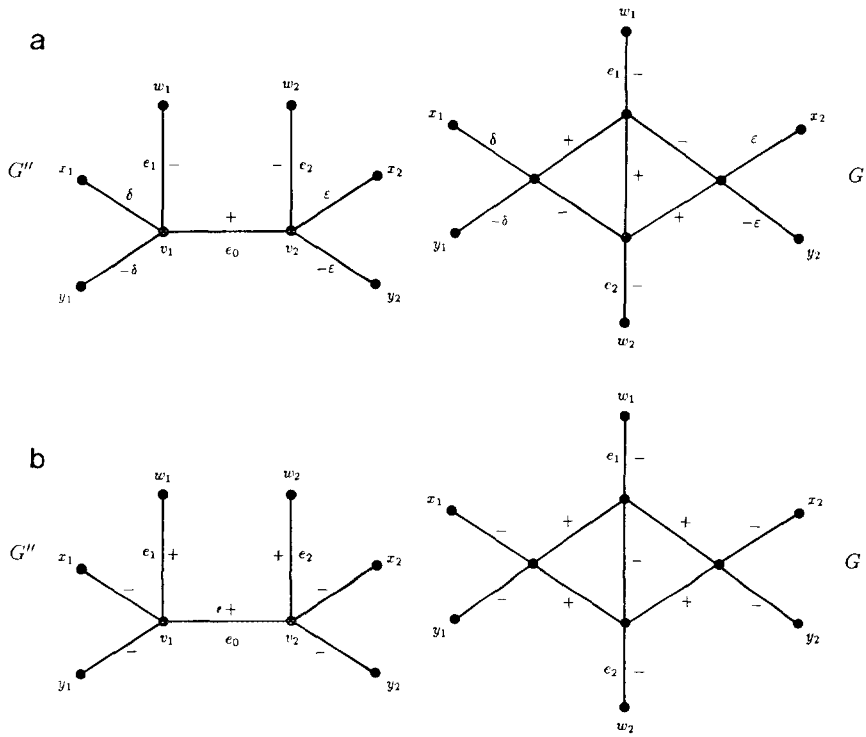

FIGURE 8

Subcase 1.2. v_1, v_2 are adjacent (i.e., the subgraph U induced by u_1, u_2, v_1, v_2 is a K_4). Let x_i (resp. y_i) be the unique neighbour of u_i (resp. v_i) not in U, $i=1, 2$. Some of the vertices x_1, x_2, y_1, y_2 may coincide.

We concider first the case where x_1, x_2, y_1, y_2 are distinct. The argument here runs along the same lines as in Subcase 1.1. Form G' by contracting U to a single vertex u. Obviously, G' is connected and a 4,2'-graph. Note that for degree reasons none of the four vertices of U is a cut-vertex of G. Hence the cut-vertices of G' are those of G, and possibly u, the latter if and only if some pair of edges e_i, e_j forms a cut of G, where $e_1 = [u_1, x_1]$, $e_2 = [u_2, x_2]$, $e_3 = [v_1, y_1]$, $e_4 = [v_2, y_2]$.

If u is not a cut-vertex of G', then any triangle-free balanced bicoloration of G' lifts to G as in Fig. 9.

If u is a cut-vertex of G', then w.l.o.g. $\{e_1, e_3\}$ is a cut of G. Instead of G' we consider the 4,2'-graph $G'' := G\backslash F$, where F is the 4-cycle $u_1 v_1 u_2 v_2$. $G\backslash\{e_1, e_3\}$ consists of exactly two components Q, Q', one of which, say Q, contains U. $Q\backslash F$ is connected, otherwise it would consist of two

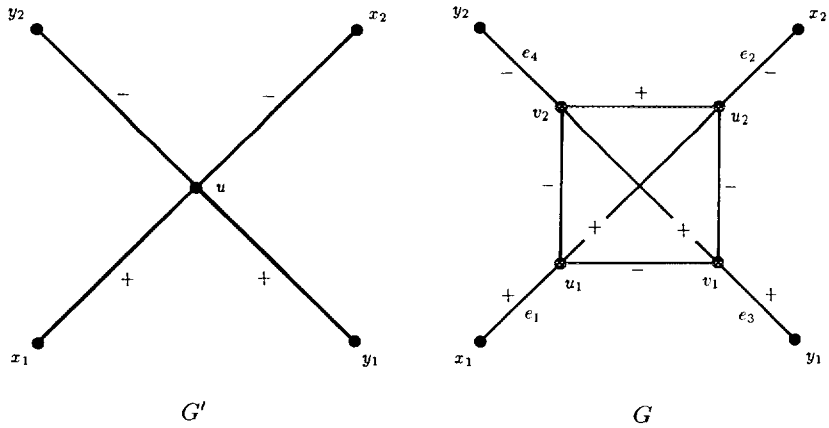

FIGURE 9

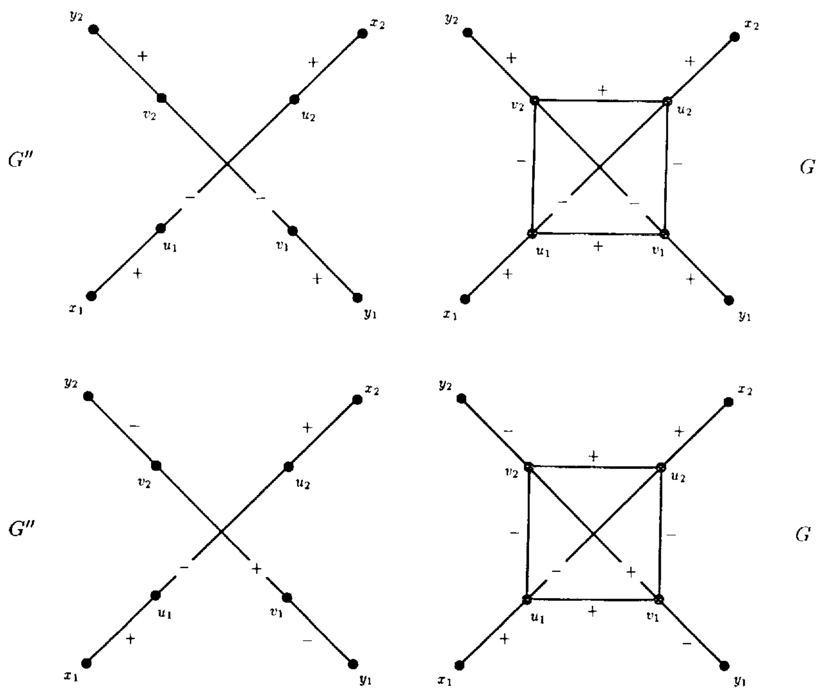

FIGURE 10

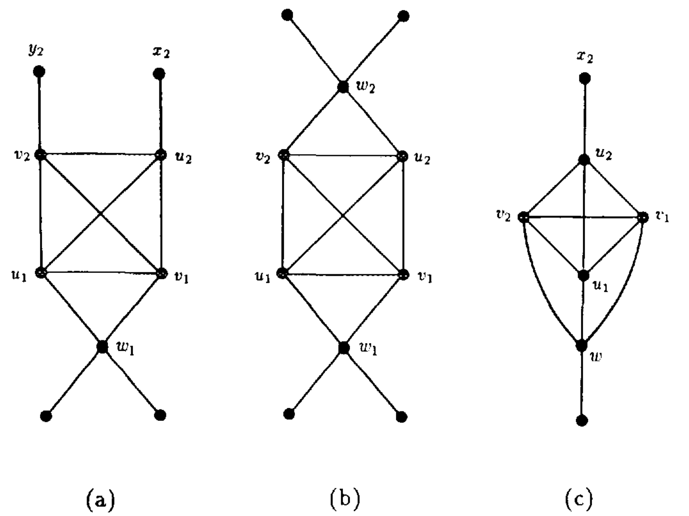

FIG. 11. (a) $w_1 := x_1 = y_1$; (b) $w_i := x_i = y_i$, $i = 1, 2$; (c) $w := x_1 = y_1 = y_2$.

components with unique vertices of odd degree, viz. u_1 and v_1. Hence G'' contains a cycle passing through all four vertices of U; i.e., none of them is a cut-vertex of G''. Thus G'' has the same cut-vertices as G.

Bicolorations of G'' can be lifted to G as shown in Fig. 10. Clearly, balance and triangle-freeness are preserved.

When some of the vertices x_1, x_2, y_1, y_2 coincide we obtain the configurations shown in Figs. 11a, b, c. (If $x_1 = x_2 = y_1 = y_2$ the result is K_5, but in K_5 every edge is in three triangles, so this case cannot arise.) For

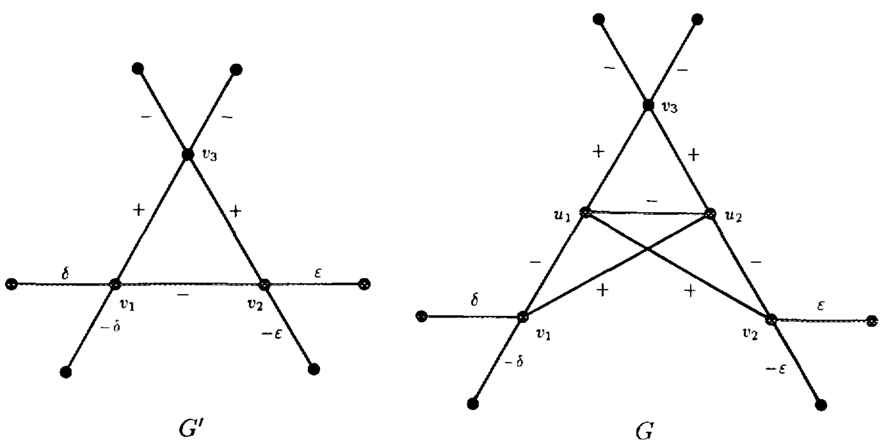

FIGURE 12

cases (a) and (b) we take $G' := G \setminus F$ with $F = u_1 v_1 u_2 v_2$. In case (c), G' is obtained by removing u_1, v_1, v_2 and their incident edges, and adding the edge $[u_2, w]$. That G' has the required properties and that bicolorations can be properly lifted is straightforward; we leave the details to the reader.

Case 2. Each edge of G belongs to at most one or to exactly three triangles. Let $e = [u_1, u_2]$ be an edge contained in three triangles $\Delta_i = u_1 u_2 v_i$, $i = 1, 2, 3$. If v_1, v_2 are adjacent in G and v_1, v_3 are non-adjacent, then $[u_1, v_1]$ is an edge belonging to exactly two triangles, contrary to hypothesis. This means that the subgraph U of G induced by u_1, u_2, v_1, v_2, v_3 is either $\Delta_1 \cup \Delta_2 \cup \Delta_3$ or the complete graph K_5. In the latter case $G = K_5$, since K_5 is 4-regular and G has maximum degree 4. K_5 has a factorisation into two pentagons.

If $U = \Delta_1 \cup \Delta_2 \cup \Delta_3$, consider the graph G' formed by adding to $G \setminus U$ the triangle $v_1 v_2 v_3$. G' is a connected 4,2'-graph whose cut-vertices are essential and coincide with those of G. Any triangle-free balanced bicoloration of G' can be lifted to G as shown in Fig. 12 (where δ, $\varepsilon = +1$ or -1). ∎

4. Proof of Theorem 1.3

In view of Lemma 3.1 we shall assume throughout this section that G is a *connected* 4,2'*-graph* such that

(i) *the triangles of G are mutually edge-disjoint*; and

(ii) *G has at most two essential cut-vertices* (it will not be necessary, however, to assume that all cut-vertices are essential).

We have to construct a triangle-free balanced bicoloration of G.

Step 1. Denote by H the *triangular part* of G, i.e., the union of all triangles in G. By (i), H is a 4,2-graph (in general disconnected). Since G is a 4,2'-graph, no 2-vertex of G is in H; therefore the 2-vertices of H are precisely the vertices in $Q := V(H) \cap V(G \setminus H)$.

Claim 1. *H is the line-graph of a triangle-free* 3,1*-graph.* Clearly any triangle is the line-graph of three edges forming a $K_{1,3}$. If two triangles Δ, $\Delta' \subset H$ share a vertex, then by (i) this vertex is unique, and $\Delta \cup \Delta'$ is the line-graph of two copies of $K_{1,3}$ sharing an edge. As the maximum degree of H is 4, we thus easily construct a graph I whose line-graph is isomorphic to H. Edge-disjointness of the triangles in H implies that I is triangle-free.

Throughout the remainder of the proof we shall denote by x_a the image of $a \in E(I)$ under the isomorphism $L(I) \to H$. In other words, x_a is *a* "viewed as a vertex of H."

Note that the pendant edges of I, viewed as vertices of H, are exactly the 2-vertices of H. Furthermore, the 3-vertices of I are in one–one correspondence with the triangles of H.

Step 2. Let $G_1, \ldots, G_r$, $r \geqslant 0$, be the components of the *triangle-free part* $G \backslash H$ and put $Q_i := V(G_i) \cap V(H)$, $i = 1, \ldots, r$. Since G is a 4,2′-graph we have that

$$m_i \equiv q_i \pmod 2, \tag{1}$$

where $m_i := |E(G_i)|$ and $q_i := |Q_i|$.

Denote the vertices in Q_i by $u_{i1}, \ldots, u_{iq_i}$, $i = 1, \ldots, r$. In view of the fact that the pendant edges of I and the vertices in Q are the same thing, there is a natural one–one correspondence which attaches to each $u_{ij} \in Q$ a 1-vertex v_{ij} of I.

We now extend I to a 3-regular graph J (spanned by I) as follows. If $r = 0$, i.e., if $H = G$, then I is 3-regular, and we take $J := I$. If $r > 0$, J is formed by adding $q_1 + \cdots + q_r$ new edges to I in such a way that for each $i = 1, \ldots, r$, the vertices $v_{i1}, v_{i2}, \ldots, v_{iq_i}$ (in some order) form a q_i-cycle C_i. Clearly $C_1, \ldots, C_r$ are *induced* cycles of J. Note that $q_i = 1$ means that C_i is a loop attached at the vertex v_{i1}; if $q_i = 2$, C_i is a digon. However, the fact that J may not be a simple graph is irrelevant in the context of Petersen's theorem, which we now apply. Because of the connectedness of G, J is connected (even when H is disconnected).

Claim 2. *J has at most two bridges.* The edges of the cycles $C_1, \ldots, C_r$, i.e., the edges of $J \backslash I$, obviously are not bridges of J. As to the edges of I (which correspond to the vertices of H), note that the line-graph $L(J)$ contains (an isomorphic copy of) H as an induced subgraph. Bridges of J and cut-vertices of $L(J)$ are the same thing, and any cut-vertex of $L(J)$ which belongs to H is a cut-vertex of H. On the other hand, any essential cut-vertex of G is either of degree 2 in H or it is a cut-vertex of H. Since by assumption G has at most two essential cut-vertices, Claim 2 follows.

Step 3. We now construct a bicoloration of G, starting with $G \backslash H$. By Petersen's theorem, J has a perfect matching M. For $i = 1, \ldots, r$ let M_i be the set of all edges in M having exactly one endpoint in C_i and denote by P_i the corresponding set of vertices in H. In the notation introduced earlier, $P_i = \{x_a : a \in M_i\}$. The edges in M_i are pendant edges of I; hence $P_i \subset Q_i$. Note that $p_i \equiv q_i \pmod 2$, where $p_i := |P_i|$. Therefore by (1),

$$m_i \equiv p_i \pmod 2, \quad i = 1, \ldots, r. \tag{2}$$

To obtain a bicoloration of $G \backslash H$ we colour each component G_i separately, using a slight modification of Petersen's classical method for the

2-factorisation of 4-regular graphs. Take an arbitrary eulerian walk on G_i and colour the edges of G_i alternatingly $+1$ and -1 in the order in which they occur in the walk, except when the common vertex of two successive edges belongs to P_i, in which case both edges receive the same colour. Because of (2) this procedure results in a well-defined bicoloration of G_i and the balanced vertices of G_i are precisely those which are not in P_i. Combining the bicoloration of the various components we obtain a bicoloration of $G \backslash H$ for which the only unbalanced vertices are those in $P_1 \cup \cdots \cup P_r$.

Step 4. To colour the edges of H we employ the same procedure as in the proof of Theorem 1.2. Given a non-pendant edge a of I (so that x_a is a 4-vertex of H) let D_a be the double triangle in H formed by the two triangles Δ_a, Δ'_a whose common vertex is x_a. Denote by M' the set of all edges in M which are non-pendant in I. Colour each D_a, $a \in M'$, as indicated in Fig. 3 (it does not matter which of the two triangles Δ_a, Δ'_a gets $+1$ as the dominant colour; cf. the remark at the end of Section 2). The D_a's, $a \in M'$, are mutually edge-disjoint; hence we obtain a well-defined bicoloration of $H_M := \bigcup \{D_a : a \in M'\}$ which is obviously balanced and triangle-free. Note that H_M contains all the triangles in H which do not share a vertex with $G \backslash H$.

It remains to colour $H \backslash H_M$, that is to say, the triangles in H which, viewed as vertices of I, are the vertex of attachment of an edge in $M \backslash M'$, i.e., a pendant edge of I belonging to M. Given such a triangle Δ, let $a \in M$ be the corresponding pendant edge. Then $a \in M_i$ for some i, $1 \leqslant i \leqslant r$. Returning to H, the vertex x_a belongs to P_i. Hence the two edges of $G \backslash H$ incident with x_a have already been coloured and have the same colour, say ε_a (Step 3). Now colour Δ by assigning the colour $-\varepsilon_a$ to the two edges incident with x_a, and ε_a to the third edge. Thus none of the triangles in $H \backslash H_M$ is monochromatic, i.e., the bicoloration of G obtained by combining the bicolorations of $G \backslash H$, H_M, and $H \backslash H_M$, is triangle-free. ∎

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